ON NUMERICAL RANGES AND ELEMENTARY OPERATORS

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Declaration

Declaration by the Candidate

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Dedication

I dedicate this thesis with much love and esteem to my family.

Abstract

The numerical range of a bounded linear operator on a Hilbert space H, is the range of the restriction to the unit sphere of the quadratic form associated with the operator. An elementary operator is a bounded linear mapping on the set of bounded linear operators acting on an infinite dimensional complex Hilbert space. Properties of elementary operators have been investigated during the past three decades under a variety of aspects such as their spectra, compactness, norm properties, numerical range among others. However, through all these studies it emerges that, for a general elementary operator, a precise description of its properties has not been explored exhaustively. Thus a generalized description of these properties on the various generalizations of numerical ranges of an elementary operator is missing and hence have been studied in this research work. The general objective was to establish the relations that exists between the numerical range of the elementary operator and that of the implementing operators as the operator acts on the various algebras. Specifically, the objectives were: to establish some of the properties of the numerical range that hold for an elementary operator acting the algebra $L(H)$ and to determine the relationship that exists between the elementary operators and numerical ranges of their implementing operators both in a normed ideal and in a Hilbert space considered as a $C*-$algebra. In particular, the convexity of the algebra numerical range was shown as well as its equality to the algebra numerical range of the left and right multiplication operators. The algebra numerical range of a generalized derivation restricted to a norm ideal was established to be equal to the set difference of the algebra numerical ranges of the implementing operators. Finally, the closed convex hull of the maximal numerical range of the implementing operators was shown to be contained in the algebraic maximal numerical range of an elementary operator restricted to an operator algebra. Working from the known to the unknown, we have borrowed from the already established relationships between the spectrum of an elementary operator and the joint spectrum of two commuting n-tuples and obtained relations in terms of the numerical ranges. Another approach utilized was algebraically constructive in nature. From the theory of Banach spaces, we have for instance, the famous Hahn Banach theorem that allows us to algebraically construct functionals in a subspace and we are guaranteed of an extension in the whole space under consideration. With regards to the application of our findings, the numerical range is often used to locate the spectrum of an operator. Certain problems in quantum mechanics, for instance, approximation by commutators, the Heisenberg uncertainly principle, among others correspond to elementary operators and the findings obtained from our research will contribute to the theoretical knowledge that such physicists and applied mathematicians need.

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Notations

$\mathbb{R}$- The fied of real numbers
$\mathbb{C}$- The field of complex numbers
$\|.\|$ - The norm function
$\langle . \rangle$- Inner product function
H- An infinite dimensional complex Hilbert space
$L(H)$ -The algebra of bounded linear operators acting on H
$K(H)$- The set of compact operators
$W(A)$- Numerical range of an operator $A \in L(H)$
$w(A)$- Numerical radius of an operator $A \in L(H)$
$\sigma(A)$- The spectrum of the operator A
$W_{\circ}(A)$ -The maximal numerical range
$\mathscr{A}$- A complex algebra
$V\,(a; \mathscr{A})$- The algebraic numerical range
$W_e(T)$ - The essential numerical range
$V_{\circ}(a, \mathscr{A})$ -The algebraic maximal numerical range
$essW_{\circ}(A)$ -The essential maximal numerical range
$S(\mathscr{A})$- The set of all states
$R_{A,B}$ - The elementary operator associated with operators A and B.
For subsets $\sigma, \tau \subset \mathbb{C}^n$, $\sigma \circ \tau = \{\alpha \circ \beta \equiv \sum \alpha_i \beta_i : \ \alpha = (\alpha_1, ..., \alpha_n) \in \sigma, \beta = (\beta_1, ..., \beta_n) \in \tau\}$

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Chapter 1

INTRODUCTION

1.1 Background Information

In the theory of matrices, the concept of a quadratic form associated with a matrix and its applications is quite well known. A natural extension of these ideas in finite and infinite dimensional spaces leads to the concept of numerical range and most quadratic questions about an operator revolves around this range. The numerical range of an operator when considered on finite dimensional spaces is sometimes referred to as field of values, a term commonly used in matrix theory. Other terms that have been used in place of numerical range include Wertovorrat, Hausdorff domain, range of values among others. However, the first two have outlived the rest but the advantage goes decisively to the first, that is, numerical range and its mostly preferred by operator theorists. The core of numerical range theory lies on the Hilbert space setting. In fact, in the early studies of Hilbert space by researchers such as Hilbert, Hellinger, Toeplitz, and others, the objects of chief interest were quadratic forms. For a bounded linear operator T on a Hilbert space H, we may associate to it a sesquilinear form

φ_T given by$\varphi_T(x, y) = \langle Tx, y\rangle\,, \quad x, y \in H$

and the corresponding quadratic form

$\widehat{\varphi_T}(x) = \varphi_T(x, x) = \langle Tx, x\rangle\,, \quad x \in H.$

With this in mind, the numerical range as defined below is simply the range of the restriction of $\widehat{\varphi_T}$ to the unit sphere. Formally we define the numerical range implemented by the operator T by

$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$.

Originally, the concept of numerical range was initiated by Toeplitz and Hausdorff in 1918 for matrices, that is, in finite dimensional spaces, but this definition equally applied well to operators on infinite dimensional Hilbert spaces. Toeplitz and Hausdorff proved the most crucial property of the numerical range, that is, its convexity. Other fundamental properties include its non-emptiness, inclusion of the spectrum of T within its closure and also that it lies in the closed disc of radius $\|T\|$ centered at the origin. Ever since, motivated by theoretical study and applications, studies of the numerical range have constituted a very wide and active field of research in matrix analysis and operator theory. Indeed there is a vast, dynamic current research on these concepts and their generalizations that have been documented. In particular, this is because they are very useful in studying and understanding the role of matrices and operators in applications such as numerical analysis and differential equations. One of the beauties of the numerical range is that its properties and use extends over three areas in mathematics, these being matrix analysis, operator theory and differential equations. Generalizations of the numerical range in the finite and infinite dimension cases, in Banach spaces and Banach algebras are fully explored in Bonsall and Duncan (1973); Gustafson and Rao (1997); Horn and Johnson (2012) in detail. From the matrix analysis viewpoint, a number of variations on the numerical range have been and are currently being studied including the $k-$ numerical range; the $C-$ numerical range; the $M-$ numerical ranges and their generalizations. From the operator theory viewpoint, generalizations here have largely been the extension to Banach spaces based upon the Hahn-Banach theorem and the notion of a semi-inner product.

On the other hand, the study of elementary operators emanated from the theory of matrix equations originally done by Sylvester (1884). In a series of notes, Sylvester obtained the eigenvalues of the matrix operators corresponding to the elementary operator on the square matrices. The term elementary operator was initially adopted by Lumer and Rosenblum in the late $1950's$ as documented in Lumer and Rosenblum (1959). The two made emphasis on the spectral properties of these operators and their applications to systems of operator equations. Studies on properties of these operators have since been of great concern to many operator theory mathematicians. Among these properties include their numerical ranges, spectrum, compactness. The literature pertaining to elementary operators is by now readily available, and there are many profound survey results and expositions on certain aspects of these operators and in particular, on their spectral and structural properties. However, though much has been done, there are still many open problems particularly because their properties are often directly connected to the structure of the underlying space the operator is acting on. We note here that the inner and the generalized derivation have enjoyed a lot of attention ever since the study of elementary operators began. Their norms have been computed, their spectral properties have been characterized and even their numerical ranges established in various spaces. However, for the general elementary operator, explicit formula for their norms, spectra and numerical range has been a challenge. In terms of the space settings, for example, whether the operator is acting on a Banach space, a Banach algebra or a C* -algebra, the classical numerical range for an elementary operator in these settings has sufficiently been explored by researchers and so majority of the work currently being carried out is geared towards those generalizations of the numerical range that still remain partially explored. The maximal numerical range is such a generalization. It is Stampfli who introduced this concept and used it as the key tool to determine the norm of derivations.

1.2 Statement of the problem

Since their initiation, for the elementary operator
$R_{A,B}(X) = A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n, \forall X \in L(H)$ associated with the $n-$tuples A and B on $L(H)$ into itself, many facts about the relationship between the spectrum of the elementary operator and the spectrum of the implementing operators have been established. This is not the case in relation to how the numerical range of an elementary operator is related to the numerical range of the implementing operators, that is, $W(R_{A,B}), W(A)$ and $W(B)$. Apparently, the only elementary operators for which the various numerical ranges have been computed are the inner and the generalized derivations. Kyle (1978), for example, examines the relationship between the numerical range of an inner derivation, and that of its implementing element. Magajna (1987) gives the essential numerical range of the the generalized derivation defined on the Hilbert-Schmidt class in terms of the numerical and the essential numerical ranges of the implementing operators. Shaw (1984) in particular, established that the algebra numerical range of a generalized derivation restricted to a norm ideal J is equal to the difference of the algebra numerical ranges of the implementing operators provided that J contains all finite rank operators and is suitably normed.
Most often, the properties of an operator are derived from its domain and range. For the elementary operators, though much has been done, there is still much lacking with regards to the relations that exist between the various generalizations of numerical ranges of an elementary operator and that of the implementing operators in diverse algebras.

1.3 Justification

Despite the conceptual simplicity of the definition of numerical range as the image of the unit sphere of H under a continuous map $x \longrightarrow \langle Ax, x\rangle$, there are many unanswered questions and key issues surrounding it that still remain non explored. Some are curiosity driven and some are application driven. In fact, very little about the numerical range is obvious and more so in infinite dimensional spaces.

As noted earlier, the concept of numerical range has been generalized in different directions. One such direction, is the maximal numerical range introduced by Stampfli (1970) to derive an identity for the norm of a derivation on $L(H)$. Unlike the other generalizations, the maximal numerical range has not been largely explored by researchers as many only refer to it in their quest to determine the norm of operators. Furthermore, many facts about the relation between the spectrum of elementary operators and the spectrum of the implementing coefficients operators are already known. This is not the case for the relationships that exist between the various numerical ranges of the elementary operator and the numerical ranges of the implementing operators. We seek to establish these relations, for example, how the algebraic maximal numerical range of elementary operators is related to the closed convex hull of the maximal numerical range of the implementing operators $A = (A_1, A_2, ..., A_n)$, $B = (B_1, B_2, ..., B_n)$, on the algebra of bounded linear operators on a Hilbert space H .

1.4 Objectives

The general objective of the study was to establish the relationship between the numerical range of an elementary operator and that of the implementing elements.

Specific Objectives

The study aimed to:

i) Establish the properties of the numerical range that hold when we consider an elementary operator acting on the algebra $L(H)$.

ii) Examine the relationship between the numerical range of the restriction of a generalized derivation to a norm ideal J and that of its implementing operators.

iii) Determine how the algebraic maximal numerical range of elementary operators is related to the closed convex hull of the maximal numerical range of the implementing operators on the algebra of bounded linear operators on a Hilbert space H considered as a $C*-$ algebra.

1.5 Significance

The numerical range of an operator A contains crucial information on the properties of the operator. Even when the operator is not explicitly known, it allows one to deduce many properties of the operator. For instance, the numerical range is often used to locate the spectrum of an operator since the spectrum is known to be contained within the numerical range of the operator. Given the numerical range of an operator, one is also capable of making deductions on the properties of the operator both algebraic and analytic . The upper and lower norm bounds, dilations with simple structure, among others can also be obtained given the numerical range. Furthermore, the geometric properties of the numerical range are used to classify special types of operators, for example, the self adjoint, normal, unitary among others.

It is clear from the definition of the numerical range and numerical radius that these two are intimately related. The numerical radius of A and the distance of $W(A)$ to the origin are used in studying perturbation, stability, convergence and approximation problems. In particular, as an example, very often, the numerical radius has been used as a reliable indicator for rate of convergence of iterative methods. It also plays a crucial role in the stability analysis of finite difference approximations of solutions to hyperbolic initial value problems. Furthermore, numerical radius has recently been associated with stability issues of Hermitian generalized eigen-problems and of higher order dynamical systems. In engineering, the numerical range is used as a rough estimate of eigenvalues of an operator.

One of the most important operator equation in terms of application is of the form $AX - XB = Y$ and hence corresponds to an elementary operator.

There are problems in quantum mechanics, for example, approximation using the commutators $AX-XA$ or by $AX-XB$ that have also aroused much curiosity for researchers in pure mathematics, and specifically in matrix and operator theory. Formulated mathematically, the Heisenberg uncertainly principle corresponds to saying that there exists a pair (A, X) of linear transformations and a non-zero scalar α for which $AX - XA = \alpha I$. Clearly, these correspond to elementary operators of length two and the findings obtained from our research will contribute to the theoretical knowledge that such physicists and applied mathematicians need.

Chapter 2

THEORY

The numerical range of an operator greatly depends on the underlying space the operator is acting on. The many generalizations that have been studied mostly differ depending on whether, for example, the set under consideration is finite or infinite dimension, a Banach space, Hilbert space, Banach algebra or a C*-algebra. In this chapter, the necessary theoretical background material from the theories of normed spaces, Banach algebras and C*-algebras are given that are of relevance in the study.

2.1 Banach Algebras

Definition 2.1. An algebra is a vector space $\mathscr{A}$ equipped with a bilinear product.

A sub-algebra of $\mathscr{A}$ is a vector subspace $\mathscr{B}$ that is closed under multiplication, that is, $b, b' \in \mathscr{B} \Rightarrow bb' \in \mathscr{B}$.

A norm function $\|.\|$ defined on $\mathscr{A}$ is said to be sub-multiplicative if

$\|ab\| \leq \|a\| \|b\|$ for all $a, b \in \mathscr{A}$. A normed algebra is an algebra with a sub-multiplicative norm defined on it. If a normed algebra $\mathscr{A}$ possesses a unit element, say e such that $ae = ea = a$, for all $a \in \mathscr{A}$, then $\mathscr{A}$ is referred to as a unital normed algebra. A complete normed algebra is called a Banach algebra and a complete unital normed algebra is a unital Banach algebra.

If a normed or a Banach algebra has no unit element, we can adjoin a unit to it.

The unitization of a normed algebra $\mathscr{A}$ over a field $\mathbb{F}$, denoted by $\mathscr{A}+\mathbb{F}$, is the normed algebra consisting of the set $\mathscr{A}\times\mathbb{F}$ with addition, scalar multiplication and product defined by:

$(a,\alpha)+(b,\beta)=(a+b,\alpha+\beta)$

$\lambda(a,\alpha)=(\lambda a,\lambda\alpha)$,

$(a,\alpha)(b,\beta)=(ab+\alpha b+\beta a,\alpha\beta)$, for all $a,b\in\mathscr{A},\alpha,\beta\in\mathbb{F}$

The unit element in $\mathscr{A}+\mathbb{F}$ is (0,1). $\mathscr{A}+\mathbb{F}$ is a normed algebra with the norm defined by $\|(a,\alpha)\|=\|a\|+|\alpha|$

Some important examples of normed and Banach algebras are give here below:

Example 2.1. Let X be a normed linear space. Then $L(X)$, the set of all bounded linear operators on X, with the point-wise defined operations for addition and scalar multiplication, composition for the product and the operator norm is a normed algebra. If X is a Banach space, then $L(X)$ is complete and is thus a Banach algebra. The identity operator I is the unit element of $L(X)$. Composition of maps is a non-commutative operation and so $L(X)$ is an example of a non-abelian Banach algebra.

Example 2.2. The scalar field $\mathbb{C}$ of complex numbers, with the usual multiplication and the absolute value function as the norm is a Banach algebra.

Example 2.3. If S is a set, the set of all bounded complex valued functions on S, denoted by $\ell^\infty(S)$, is a unital Banach algebra with the point-wise defined operations for addition, scalar multiplication and product of functions. The norm here is the sup-norm.

Example 2.4. Let X be a topological space and $C_b(X)$ the collection of all bounded continuous functions on X. Then $C_b(X)$ is a closed sub-algebra of $\ell^\infty(X)$. Hence, $C_b(X)$ is a unital Banach algebra.

If X is a compact topological space, $C(X)$, the set of all continuous functions from X to $\mathbb{C}$, is equal to $C_b(X)$ since any continuous function on X is bounded.

A continuous function f on a locally compact Hausdorff space X is said to vanish at infinity if the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$. The collection of all such functions is denoted by $C_0(X)$. It is a closed sub-algebra of C_bX and hence a Banach algebra. $C_0(X)$ is unital if and only if X is compact and $C_0(X)$ coincides with $C(X)$.

For $f \in C(X)$, the support of f, denoted by $suppf$, is the closure of the set: $\{x \in X : f(x) \neq 0\}$.

From the definition, we deduce that the support of a function is always a closed set. If this set is compact, then we say that f has compact support. Denote by $C_c(X)$, the collection of all compactly supported continuous functions on X. Then, $C_c(X)$ is a dense sub-algebra of $C_0(X)$.

The Banach algebra $C_0(X)$ is one of the most significant examples of abelian Banach algebra.

A left ideal in an algebra $\mathscr{A}$ is a non empty vector subspace J of $\mathscr{A}$ such that for all $a \in \mathscr{A}, b \in J, ab \in J$. Similarly, for a right ideal, $ba \in J$ for every $a \in \mathscr{A}, b \in J$. A vector subspace $J \subseteq \mathscr{A}$ that is both a left and a right ideal in $\mathscr{A}$ is called a two sided ideal or simply an ideal.

Example 2.5. Suppose $\mathscr{A}$ is a normed algebra, J a two sided closed ideal in $\mathscr{A}$. Then we can form $\mathscr{A}/J$, the quotient algebra with addition, multiplication and scalar operations defined as follows: $\forall\ a + J, b + J \in \mathscr{A}/J,\ \lambda \in \mathbb{C}$, then

$(a + J) + (b + J) = (a + b) + J$,

$(a + J)(b + J) = ab + J$ and

$\lambda(a + J) = \lambda a + J$ respectively.

$\mathscr{A}/J$ is a normed algebra when we endow it with the quotient norm:

$$\|a\| = \|a + J\| = inf\,\{\|a + k\| : k \in J\}\,.$$

If $\mathscr{A}$ is complete in the norm, then so is $\mathscr{A}/J$.

2.2 C*- Algebras

Definition 2.2. Let $\mathscr{A}$ be a complex algebra. A conjugate linear map $* : \mathscr{A} \to \mathscr{A}$ is called an involution on the algebra $\mathscr{A}$ if it satisfies the axioms below:

(i) $(a^*)^* = a$ for all $a \in \mathscr{A}$.

(ii) $(ab)^* = b^*a^*$ for all $a, b \in \mathscr{A}$.

(iii) $(\alpha a + \beta b)^* = \overline{\alpha}a^* + \overline{\beta}b^*$.

for all a and b in $\mathscr{A}$ and α , β in $\mathbb{C}$.

The pair $(\mathscr{A}, *)$ is called an involution algebra, or a $*$-algebra.
An element $a \in \mathscr{A}$ is self-adjoint or hermitian if $a = a^*$. If S is a sub-algebra of $\mathscr{A}$, then $S^* = \{a^* : a \in S\}$ and if $S^* = S$, then S is self adjoint. Furthermore, $a \in \mathscr{A}$ is said to be normal if it commutes with its adjoint, that is, $a^*a = aa^*$ and it is unitary if $a^*a = aa^* = e$
A Banach $*$-algebra is an involution algebra $\mathscr{A}$ endowed with a complete sub-multiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathscr{A}$.
Moreover, if $\mathscr{A}$ possesses a unit element e such that $\|e\| = 1$, we call $\mathscr{A}$ a unital Banach $*$-algebra. A norm which is such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathscr{A}$ is referred to as a C^*- norm. A C^*-algebra is a Banach $*$-algebra that is complete in the C^*- norm. A C^*-algebra $\mathscr{A}$ is deemed to be unital or to have a unit e if it has an element, denoted by e, satisfying $ea = ae = a$ for all a in $\mathscr{A}$. For a nontrivial unital C^*-algebra with unit e, then automatically $e^* = e$ and $\|e\| = 1$. If a C^*-algebra is non unital, it can always be unitized by adjoining a unit to it. A subset S of a C^*-algebra is called C^*-sub-algebra if it is a C^*-algebra with the inherited operations, involution and norm.

Example 2.6. The scalar field $\mathbb{C}$ of complex numbers is a unital C^*-algebra with complex conjugation $\lambda \mapsto \overline{\lambda}$ as the involution map.

Example 2.7. Let S be a set and $\ell^\infty(S)$ the set of all bounded complex valued functions on S, then $\ell^\infty(S)$ is a C^*-algebra with involution given by $f \mapsto \overline{f}$.

Example 2.8. Let Ω be a locally compact Hausdorff space. Then $C_0(\Omega)$ is a C^*-algebra with involution $f \mapsto \overline{f}$. Also, the set $C_0(\Omega, \mathscr{A})$, of all continuous functions $f : \Omega \longrightarrow \mathscr{A}$ vanishing at infinity is a non-unital commutative C^*-algebra.

Next we recall some few key results on positive elements in a C^*-algebra.
Let $\mathscr{A}$ be a C*-algebra. An element $a \in \mathscr{A}$ is said to be positive if a is hermitian and $\sigma(a) \subseteq \mathbb{R}^+$, $\sigma(a)$ the spectrum of a. Equivalently, $a \in \mathscr{A}$ is positive if $a = b^2$ for some self-adjoint $b \in \mathscr{A}$ or $a = c^*c$ for some $c \in \mathscr{A}$. The positive elements of $\mathscr{A}$ form a cone $\mathscr{A}^+$ in that, if a, b are positive elements, then their sum $a + b$ and the scalar multiple λa are positive also, for all $\lambda \in \mathbb{R}^+$.
A linear map $f : \mathscr{A} \to \mathscr{B}$ between two C*-algebras is said to be positive if f maps positive elements in $\mathscr{A}$ to positive elements in $\mathscr{B}$, that is, $f(\mathscr{A}^+) \subset \mathscr{B}^+$.
For example, every $*$-homomorphism $\phi : \mathscr{A} \to \mathscr{B}$ is a positive map.
If the co-domain $\mathscr{B}$ is the scalar field $\mathbb{C}$, then such a positive linear map $f : \mathscr{A} \to \mathbb{C}$ is called a positive linear functional. If, in addition, f is bounded and of unit norm with $f(e) = 1$, e the identity element, then f is called a state. The term "state" is borrowed from mathematical physics. The observables of a physical system correspond to the self-adjoint elements of a C^*- algebra and the value $\omega(a)$ is supposed to be the expected value of the observable a in the "state" ω. To know the expected values of the observables of the system is to know the "state" of the system.

Example 2.9. Let Ω be a compact Hausdorff space and μ the probability measure on Ω. Define $\psi : C(\Omega) \longrightarrow \mathbb{C}$ by
$\psi(f) = \int_\Omega f(x)\ d\mu(x)$ for all $f \in C(\Omega)$.
ψ as defined is a state.

Example 2.10. Let $\mathscr{A}$ be a C^*-algebra and $\pi : \mathscr{A} \longrightarrow B(H)$ be a $*$homomorphism. For all $x \in H$, define $\psi_x : \mathscr{A} \longleftrightarrow \mathbb{C}$ by

$\psi_x(a) = \langle \pi(a)x, x \rangle$ for all $a \in \mathscr{A}$.

ψ_x is a positive linear functional of norm one and hence a state.

The set of all states is referred to as the state space, usually denoted by $S(\mathscr{A})$. $S(\mathscr{A})$ is a closed subset of the unit ball of the dual $S(\mathscr{A})^*$ in the weak-$*$ topology, hence compact by the Banach-Alaoglu Theorem. $S(\mathscr{A})$ is non-void since by the Hahn-Banach theorem, there exists $f \in \mathscr{A}^*$ such that $f(e) = 1 = \|f\|$. It is also a convex set.

A representation of a C^*- algebra $\mathscr{A}$ is a pair (π, H) where H is a Hilbert space and $\pi : \mathscr{A} \longrightarrow L(H)$ is a $*$-homomorphism. A representation is called faithful if π is injective.

Theorem 2.1. *(Gelfand-Naimark-Segal Theorem, Murphy (2014))*
If $\mathscr{A}$ is any C^*- algebra, then there exists a Hilbert space H and a faithful representation $\pi : \mathscr{A} \longrightarrow L(H)$.

The proof of this theorem can be found in Murphy (2014). C*-algebras are closely connected to operators on a Hilbert space in such a way that if H is a Hilbert space, $L(H)$ the set of all bounded linear operators on H, is a C*-algebra with its usual operator norm and the adjoint operation as the involution. By the above celebrated Gelfand-Naimark-Segal theorem, every abstract C*-algebra can be thought of as a C*-sub-algebra of $L(H)$ for some Hilbert space H. Defining abstract C*-algebras this way is convenient since it allows many operations like quotients, direct sum, products and tensor products. The interconnection is also very fundamental as it is partly due to this concrete realization of the C*-algebras that their theory is easily accessible in comparison with more general Banach algebras. Just like it is easy to work with Hilbert spaces in contrast to general Banach spaces, the same is true of C*-algebras compared with general Banach algebras.

2.3 Compact Operators and the Calkin Algebra

A subset X of a Hilbert space H is called compact if every sequence $\{f_n\}$ from X has a subsequence which converges in X. A subset X of H is called pre-compact if the closure of X is compact.

A bounded operator $K \in L(H)$ is said to be compact if K maps bounded subsets of H into pre-compact subsets of H. Equivalently, $K \in L(H)$ is compact if it maps every bounded sequence $\{x_n\}$ of vectors in H onto the sequence $\{Kx_n\}$ which has a convergent subsequence. The class of all compact operators is often denoted by $K(H)$. For example, every operator of finite rank is compact since all balls are pre-compact in a finite dimensional space. The set of compact operators is a norm-closed , two sided, $*-$ideal in $L(H)$. If H is an infinite - dimensional complex separable Hilbert space, $K(H)$ the ideal of all compact operators acting on the Hilbert space H, then $L(H)/K(H)$ is a quotient algebra usually referred to as the Calkin algebra. For $T \in L(H)$ we define the essential norm by

$$\|T\|_e = \|T + K\ (H)\| = inf\left\{\|T + K\| : K \in K(H)\right\}.$$

For a positive bounded self-adjoint operator A on a Hilbert space H, the trace of A is defined by

$$trA = \sum_j \langle e_j, Ae_j\rangle,$$

where e_j form an orthonormal basis .

A bounded linear operator $K : H \to H$ is called a Hilbert-Schmidt operator if $trace(K^*K)$ is finite.

The Hilbert - Schmidt norm is normally defined by:

$$\|K\|_2 = \sqrt{tr(K^*K)} = \sqrt{\sum_j \|Ke_j\|^2}.$$

This definition does not depend on the choice of the basis. The Hilbert-Schmidt norm is also called the Frobenius norm. The class of all Hilbert-Schmidt operators from a Hilbert space H_1 to a Hilbert space H_2 is denoted by $\mathscr{C}^2(H_1, H_2)$ and $\mathscr{C}^2(H) = \mathscr{C}^2(H, H)$ in case $H_1 = H_2$. A Hilbert-Schmidt operator is a compact operator and hence every operator of finite rank is a Hilbert-Schmidt operator. We note here that set $K(H)$ is dense in $\mathscr{C}^2(H)$.
$(\mathscr{C}^2(H), \|.\|_2)$ is a separable Hilbert space with inner product

$$\langle A, B\rangle_2 = \sum_j \langle Ae_j, Be_j\rangle\,,$$

where e_j is an orthonormal basis on H.
$\mathscr{C}^2(H)$ is also an operator ideal in $L(H)$, that is $L(H)\mathscr{C}^2(H)L(H) \subseteq \mathscr{C}^2(H)$.
Generally, if H_1 and H_2 are separable Hilbert Spaces and $T \in L(H_1, H_2)$, for $p \in [1, \infty)$, we define the Schatten p-norm of T as:

$$\|T\|_p = \left(\sum_{n\leq 1} S_n^p\,(T)\right)^{\frac{1}{p}} = \left(tr(T^*T)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

for $S_1(T) \geq S_2(T) \geq ... \geq S_n(T) \geq ... \geq 0$, the singular values of T.
We observe that:

- $\|T\|_p^p = tr(|T|^p)$.
- $\|T\|_1$ corresponds to the trace class norm.
- $\|T\|_2$ corresponds to the Hilbert -Schmidt norm.
- $\|T\|_\infty$ is the operator norm.

For any particular operator which possesses these norms, the norms are equivalent. An operator which has a finite $p - th$ Schatten norm is a $p-$ Schatten operator.

2.4 Generalizations of the Numerical Range

The concept of numerical range has been generalized in different directions. The following are some of the generalizations relevant to this study.

Definition 2.3. Let H be a complex separable Hilbert space, and let $L(H)$ be the set of all bounded linear operators on H into itself. The **numerical range** of an operator $A \in L(H)$ is the subset of complex numbers, given by
$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$.
The numerical radius $w(A)$ of $A \in L(H)$ is defined by

$$w(A) = sup\left\{|z|, z \in W(A)\right\}.$$

Definition 2.4. Let $K(H)$ be the set of all compact operators on a Hilbert space H. The essential numerical range of $T \in L(H)$ is defined by

$$W_e(T) = \cap\left\{\overline{W(T+K)} : K \in K(H)\right\}.$$

Definition 2.5. Let $A = (A_1, A_2, ..., A_n)$ be an n-tuple with $A_i \in L(H)$ for $1 \leq i \leq n$. The **joint numerical range** of $A \in L(H)$ is given by

$$W(A) = \{(\langle A_1x, x\rangle, \langle A_2x, x\rangle, ..., \langle A_nx, x\rangle) : x \in H, \|x\| = 1\}$$

Definition 2.6. Let $L(X)$ be the Banach algebra of all bounded linear operators acting on a complex Banach space. For $A \in L(X)$, the **spatial numerical range** is defined by:
$W(A) = \{\langle Ax, x^*\rangle : x \in X, x^* \in X^*, \|x\| = \|x^*\| = 1 = \langle x, x^*\rangle\}$.

Definition 2.7. The **maximal numerical range** of $A \in L(H)$, denoted by $W_\circ(A)$ is the set

$$W_\circ(A) = \left\{\lambda : \langle Ax_n, x_n\rangle \to \lambda, \|x_n\| = 1, \|Ax_n\| \to \|A\|\right\}.$$

Definition 2.8. The **Joint maximal numerical range** of $A = (A_1, A_2, ..., A_n) \in L(H)$ is given by

$$W_\circ(A) = \{\{\lambda_i\} \in \mathbb{C}^n : \langle A_i x_n, x_n\rangle \to \lambda_i, \|x_n\| = 1, \|A_i x_n\| \to \|A_i\|\}, for\ 1 \le i \le n.$$

Definition 2.9. If $\mathscr{A}$ is a C*-algebra with identity e , $a \in \mathscr{A}$ and $S(\mathscr{A}) = \{f \in \mathscr{A}^* : f(e) = 1 = \|f\|\}$, the set of states on $\mathscr{A}$.Then:

(i)The **algebraic numerical range** of an element $a \in \mathscr{A}$ is the set:

$$V(a; \mathscr{A}) = \{f(a) : f \in S(\mathscr{A})\}.$$

(ii)For $a = (a_1, ...a_n) \in \mathscr{A}^n$, the **joint algebraic numerical range** is defined by:

$$V(a; \mathscr{A}) = \{f(a_1), ..., f(a_n) : f \in S(\mathscr{A})\}.$$

Definition 2.10. $S_\circ(a, \mathscr{A}) = \left\{f : f(e) = 1 = \|f\|, f(a^*a) = \|a\|^2\right\}$, will be the set of maximal states on $\mathscr{A}$. The **algebraic maximal numerical range**, denoted by $V_\circ(a, \mathscr{A})$ is the set

$$\{f(a) : f \in S_\circ(a, \mathscr{A})\}.$$

Definition 2.11. Let $A \in L(H)$ and a be the image of A in the Calkin algebra $L(H)/K(H)$. The **essential maximal numerical range** of A, denoted by $essW_\circ(A)$, is defined to be the set:

$\{\lambda : \langle Ax_n, x_n\rangle \longrightarrow \lambda \quad where \quad \|x_n\| = 1, x_n \longrightarrow 0 \quad weakly \quad and\, \|Ax_n\| \longrightarrow \|a\|\}$.

Definition 2.12. Let $L(X)$ be the complex Banach algebra of all bounded linear operators on a complex Banach space X. For $A \in L(X)$, we define the **spatial algebra numerical range** of A by:

$\mathcal{V}(A) = \{f(Ax) : (x, f) \in \prod\}$ where $\prod = \{(x, f) \in X \times X^* : \|x\| = \|f\| = f(x) = 1\}$.

2.5 Definition of an Elementary Operator

Definition 2.13. Let $A = (A_1, A_2, ..., A_n)$, $B = (B_1, B_2, ..., B_n)$ be two n-tuples with $A_i, B_i \in L(H)$ for $1 \leq i \leq n$. The **elementary operator** $R_{A,B}$ associated with A and B is the operator on $L(H)$ into itself defined by

$$R_{A,B}(X) = A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n, \forall X \in L(H).$$

For T_1 and T_2 in $L(H)$ we have the following examples of elementary operators:

i) the left multiplication operator L_{T_1} defined by $L_{T_1}(X) = T_1X, \forall X \in L(H)$.

ii) the right multiplication operator R_{T_2} defined by $R_{T_2}(X) = XT_2, \forall X \in L(H)$.

iii) the elementary multiplication operator $M_{T_1,T_2} = L_{T_1}R_{T_2}$ defined by $M_{T_1,T_2}(X) = T_1XT_2, \forall X \in L(H)$, that is, the elementary operator of length one.

iv) the inner derivation Δ_{T_1} defined by $\Delta_{T_1}(X) = T_1X - XT_1, \forall X \in L(H)$.

v) the generalized derivation Δ_{T_1,T_2} defined by $\Delta_{T_1,T_2}(X) = T_1X - XT_2, \forall X \in L(H)$.

vi) the Jordan elementary operator, $U_{T_1,T_2}(X) = T_1XT_2 + T_2XT_1, \forall X \in L(H)$.

2.6 Other Important Definitions and Theorems

Theorem 2.2 (Banach-Alaoglu Theorem)**.** The closed unit ball B^* in X^* is compact with respect to the weak*topology.

The proof of the theorem can be found in Rudin (1991).

Definition 2.14. The **spectrum** of $A \in L(H)$, denoted by $\sigma(A)$, is defined by $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \quad is \quad not \quad invertible\}$.

Definition 2.15. An operator $A \in L(H)$ is said to be a **finite rank** operator if its range is finite dimensional. For vectors x and y in a Hilbert space H, we define a finite rank operator by $(x \otimes y)z = \langle z, y \rangle x, \ \forall z \in H$.

Definition 2.16. A set S is said to be **convex** if the line segment between any two points in S lies in S, that is, if $x, y \in S$, then $\lambda x + (1 - \lambda)y \in S, \forall \ \lambda \in [0, 1]$.Then, given any nonempty set S, there is the smallest convex set containing S denoted by $con(S)$ and is referred to as the convex hull of S. Equivalently, it is the intersection of all convex sets containing S.

Definition 2.17. A subset M of the Euclidean space is said to be **compact** if it is closed and bounded. By the Bolzano- Weierstrass theorem, this implies that any infinite sequence from the set contains a subsequence that converges to a point in the set.

Chapter 3

LITERATURE REVIEW

The concept of the classical numerical range of an operator A on a Hilbert space H was introduced by Toeplitz around 1918. The numerical range $W(A)$, also known as the field of values, as defined by Gustafson and Rao (1997), is the collection of all complex numbers of the form $\langle Ax, x\rangle$, where x is a unit vector in H. The numerical radius $w(A)$ of A is the radius of the smallest circle centered at the origin containing $W(A)$. The most important property of $W(A)$ is given by the classical Toeplitz-Hausdorff Theorem which asserts that the numerical range is a convex set, that is, if x, y are in $W(A)$, then $z = tx + (1-t)y$ is also in $W(A)$ for every real number t such that $0 \leq t \leq 1$. The following are some other basic properties of the numerical range:

i) $W(A^*) = \overline{W(A)}$.

ii) $\overline{W(A)}$ contains the spectrum of A.

iii) If $\alpha, \beta \in \mathbb{C}$, then $W(\alpha A + \beta I_H) = \alpha W(A) + \beta$.

iv) $W(U^*AU) = W(A)$ for all unitary operators (that is, $U^*U = UU^* = I$).

v) $W(A+B) \subseteq W(A) + W(B)$.

A more detailed account of the subject may be found in Bonsall and Duncan (1973, 1971); Gustafson and Rao (1997); Horn and Johnson (2012) among others.

Bonsall and Duncan devoted a lot of attention in their research work to the study of Banach algebras. Motivated by the application of the numerical range concept to the study of Banach algebras, their book Bonsall and Duncan (1973), conveniently presents detailed generalizations of the numerical range for Banach space and Banach algebra settings. If $\mathscr{A}$ is a complex unital Banach algebra and $a \in \mathscr{A}$, the algebra numerical range of a is the set; $V(a, \mathscr{A}) = \{f(a) : f \in \mathcal{P}(\mathscr{A})\}$, where $\mathcal{P}(\mathscr{A})$ is the set of all states on $\mathscr{A}$. It is known that $V(a, \mathscr{A})$ is a non empty, convex and compact set. This result follows at once from the corresponding properties of the states space being convex and weak* compact in $\mathscr{A}^*$. Since the map $f \longrightarrow f(x)$ is weak * continuous on $\mathscr{A}^*$, it follows that the range is compact and convex. It is already known that the algebraic numerical range is equal to the closure of the classical numerical range of A. This can be found in Stampfli and Williams (1968), that is, if $\mathscr{A} = L(H)$, then $V(A, L(H)) = \overline{W(A)}$.

For $a \in \mathscr{A}$, the spectrum of $a, \sigma(a) = \{\lambda \in \mathbb{C} : (a - \lambda e) \quad is \quad not \quad invertible\}$ and $\sigma(a) \subset V(a)$. The generalization of numerical range in a Banach space to the spatial numerical range was introduced by Lumer (1961). In the Banach space setting, if X is a Banach space, X may be regarded as a semi-inner-product space by choosing a function $x \longrightarrow x^*$ from X into X^* with the properties $\|x^*\| = \|x\|, \langle x, x^* \rangle = \|x\|^2$ for $x \in X$. For a bounded linear operator A on X, the (spatial) numerical range of A is the set $W(A) = \{\langle Ax, x^* \rangle : x \in X, x^* \in X^*, \|x\| = \|x^*\| = 1 = \langle x, x^* \rangle\}$. Stampfli and Williams (1968) and Lumer (1961) have showed that the closure of the convex hull of this set is equal to the algebra numerical range. In particular, when X is the Hilbert space ,$\|x\| = \|x^*\| = 1 = \langle x, x^* \rangle$ if and only if x^* is the function given by $x^* y = \langle y, x \rangle, y \in X$. Thus $W(A)$ in this case coincides with the classical numerical range.

Since every$A$ in $L(H)$ admits a decomposition $A = A_1 + iA_2$ for self adjoint $A_1, A_2 \in L(H)$, $W(A)$ can be identified with the set $\{(\langle A_1x, x \rangle, \langle A_2x, x \rangle) : x \in H, \|x\| = 1\} \subseteq \mathbb{R}^2$. This leads to the notion of joint numerical range introduced by Dekker (1969)

for an n-tuple $A = (A_1, A_2, ..., A_n)$ of commuting operators on H. The joint numerical range is essential in the study of the joint behavior of a set of operators. It is the joint numerical range that actually provides us a platform from where we study the relationship that exists between the elementary operators and its implementing operators. Unlike the usual numerical range, the joint numerical range is generally not convex for an arbitrary n-tuple of operators, however there may exist cases in which it is, for example when $n = 1$. Refer to Dash (1972) and Dekker (1969) for further details on joint numerical range. The maximal numerical range, $W_\circ(A)$ of an operator A on a Hilbert space H as defined by Stampfli (1970), is the set of all complex numbers λ for which there exists a sequence $\{x_n\}$ of unit vectors in H such that $\langle Ax_n, x_n\rangle \longrightarrow \lambda$ and $\|Ax_n\| \longrightarrow \|A\|$ as $n \to \infty$.
In the case of finite dimensional spaces, this numerical range is produced by the maximal vectors of A (an element x in H is a maximal vector for A if $\|Ax\| = \|A\| \, \|x\|$).
The set $W_\circ(A)$ has been established by Stampfli (1970) to be non-empty, closed and a convex subset of the closure of the classical numerical range $W(A)$. Unlike $W(A)$, the maximal numerical range is extremely unstable under translation as shown by Stampfli (1970). It also does not satisfy the power inequality as does $W(A)$. Stampfli (1970) used the maximal numerical range to derive the norm of the inner and generalized derivations. It was showed that $\|\Delta_A\| = 2 \; inf \, \{\|A - \lambda\| : \lambda \in \mathbb{C}\}$, and $\|\Delta_A\| = 2 \, \|A\|$ if and only if $0 \in W_\circ(A)$. Later, Fong (1979), considered the analogous concept for an element in the Calkin algebra $L(H)/K(H)$ and in a C^*- algebra $\mathscr{A}$. Here, he analogously uses the essential maximal numerical range to determine the norm of an inner derivation on the Calkin algebra. He also defined the maximal numerical range of an element in a C^*- algebra $\mathscr{A}$, that is, the algebraic maximal numerical range, $V_\circ(a, \mathscr{A}) = \{f(a) : f \in S_\circ(a, \mathscr{A})\}$, where $f \in S_\circ(a, \mathscr{A})$ is the set of maximal states. $V_\circ(a, \mathscr{A}$ is a non-empty, convex compact subset of $V(a, \mathscr{A})$, the algebra numerical range. He also gave the following key result.

Theorem 3.1. *[Fong, 1979]*
If A is an operator on a Hilbert space H, then $W_\circ(A) = V_\circ(A, L(H))$.

One of the basic properties of the numerical range is that its closure contains the spectrum of the operator. For a long time it remained a considerable challenge to compute the spectrum of a general elementary operator. Elementary operators in a more general Banach algebra context were introduced by Lumer and Rosenblum (1959), who computed the spectra of such operators and gave their applications to systems of operator equations. Apostol (1986) and Fialkow (1980, 1983, 1992) have a detailed account on the spectral and structural properties of these operators. Many other authors have subsequently studied the spectral properties of the elementary operator, with particular attention devoted to the inner and the generalized derivations. For example, in Rosenblum (1956), the spectrum of a generalized derivation was determined to be $\sigma(\Delta_{A,B}) = \{\alpha - \beta : \alpha \in \sigma(A), \beta \in \sigma(B)\}$. In 1959, Lumer and Rosenblum (1959) succeeded in extending these findings to the case of analytic elementary operators where they completely determined their spectrum in terms of the spectra of the generating operators. These results among others show that the spectral properties of the elementary operator reflect the joint spectral properties of the implementing operators.
Let $A = (A_1, ..., A_n)$ be a commuting n-tuple of operators in H. The Taylor spectrum of A is given by $\sigma_T(A, H) = \{\lambda \in \mathbb{C}^n : \ A - \lambda e \quad is \quad not \quad invertible\}$, see Taylor (1970). The relationship between the spectrum of an elementary operator and the joint spectrum (spectrum in the sense of Taylor) of two commuting n-tuples A and B has been used by researchers as a stepping stone to study other relations. It is Curto (1983) who first obtained a satisfactory formula expressing the spectrum of a general elementary operator in terms of the Taylor joint spectrum for Hilbert spaces. Here, Curto proved that, if A and B are n-tuples of commuting operators on H then:
$\sigma\left(R_{A,B}\right) = \{\sum_{i=1}^{n} \lambda_i \beta_i\}$, where $(\lambda_1, \lambda_2, ..., \lambda_n) \in \sigma_T(A), (\beta_1, \beta_2, ..., \beta_n) \in \sigma_T(B)$ where σ_T is the joint spectrum. This result was later substantially improved by

Curto and Fialkow (1987) again for Hilbert spaces.

Seddik (2002) builds on these findings to give similar results with the numerical range and the joint numerical range without the assumption of the commutativity. In particular, it was established that :

$\overline{co\{\sum_{i=1}^{n}\lambda_i\beta_i\}} \subset V(R_{A,B})$, where $(\lambda_1, \lambda_2, ..., \lambda_n) \in W(A), (\beta_1, \beta_2, ..., \beta_n) \in W(B)$, $W(A), W(B)$ the joint numerical range and $V(R_{A,B})$ the algebraic numerical range of the elementary operator. Seddik also showed this inclusion becomes an equality when $V(R_{A,B})$ is taken to be a generalized derivation, and it is strict when $V(R_{A,B})$ is taken to be an elementary multiplication operator induced by non scalar self-adjoint operators. The proof to these findings is based on the construction of a special state using the trace functional on $L(L(H))$. Seddik also showed that for the generalized derivation $\Delta_{A,B}$, the algebraic numerical range $V(\Delta_{A,B}) = V(A) - V(B)$. This was a generalization of a result earlier proved by Anderson and Foias (1975) that $V(L_A) = V(R_A) = V(A)$ for any $A \in L(H)$.

Barraa (2014) expressed the numerical range of the elementary operator in terms of the spatial numerical range of the implementing operators.

Kyle (1978) examines the relationship between the numerical range of an inner derivation, and that of its implementing operators on a complex unital Banach algebra. From the already known results on the spectra of the inner derivation, Kyle obtained the corresponding results for numerical ranges. In particular, he proved that the algebra numerical range of an inner derivation is equal to the theoretical set difference in the algebra numerical ranges of the implementing operators.

Elementary operators also induce bounded operators between operator ideals, as well as between quotient algebras such as the Calkin algebra . Over the past decades, numerous studies have been done on restrictions of elementary operators to norm ideals. Their norm properties, numerical ranges, spectra and essential spectra have been characterized. Shaw (1984) determined the numerical range of a generalized derivation on an invariant subspace δ of $L(Y, X)$ of all bounded linear operators from normed linear spaces Y and X. Shaw established that the algebra numerical range of the generalized derivation restricted to δ is equal to the difference of the algebra numerical ranges of the implementing operators, that is, $V(\Delta_{A,B}, L(\delta)) = V(A, L(X)) - V(B, L(Y))$.

Seddik (2001, 2004) considers the relationship that exists when the elementary operator acts on the Banach space of the p-Schatten class operators on H and when restricted to a norm ideal J of a complex Banach space X. The results obtained in both cases were similar. For A and B n-tuple operators on X , it was proved that :

$co\overline{\{\sum_{i=1}^{n} \lambda_i \beta_i\}} \subset V(R_{A,B} \mid J)$, where $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathcal{V}(A)$, $(\beta_1, \beta_2, ..., \beta_n) \in \mathcal{V}(B)$, $\mathcal{V}(A), \mathcal{V}(B)$, the joint spatial algebra numerical ranges of A and B, and $V(R_{A,B} \mid J)$ the algebraic numerical range of $R_{A,B}$ in a norm ideal J of $L(X)$. As a consequence of these findings, for the particular left and right multiplication operators, Seddik obtained their algebra numerical range in the norm ideal to be equal to the algebra numerical range of the implementing operator, that is, $V(L_A \mid J) = V(R_A \mid J) = V(A)$ for any $A \in L(H)$ or $A \in L(X)$. Furthermore, he showed that for the generalized derivation, $V(\Delta_{A,B} \mid J) = V(\Delta_{A,B})$. In both cases, a constructive approach is used in the proofs by using the trace function to construct a special state in the norm ideal whose extension is guaranteed by the Hahn Banach Theorem.

Elementary operators on the Calkin algebra have exhaustively been explored by Mathieu (2001). In this paper, Mathieu gives the spectral, compactness, norm and positivity properties of these operators. Magajna (1987) on the other hand has computed the essential numerical range of the restriction of a generalized derivation to the class of Hilbert-Schmidt. Here it is showed that $V_e(\Delta_{A,B}) = co\,[(V_e(A) - V(B)) \cup (V(A) - V_e(B))]$. Algebraic tensor products have extensively been employed in their work. Barraa (2005), also gave results on the essential numerical range of the restriction of an elementary operator to the class of Hilbert-Schmidt operators, from a Hilbert space H to a Hilbert space K. More precisely, for the elementary operator $R_{2,A,B} \mid C^2(H,K)$, it was showed that

$co\,[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_{2,A,B})$, where $V_e(.)$ is the essential numerical range, $W(.)$ the joint numerical range, $W_e(.)$ the joint essential numerical range and $R_{2,A,B}$ the restriction of $R_{A,B}$ to $C^2(H,K)$.

Chapter 4

METHODOLODY

4.1 Convexity

There are different ways to prove that a set C is convex. Some of the most common ones include:

- Using the definition of a convex set.
- Writing C as the convex hull of a set of points , or the intersection of a set of halfspaces.
- Building it up from convex sets using convexity preserving operations.

In this study, the third approach has been employed. The state space is known to be convex and weak$*-$ compact in the dual space. The map $f \to f(a)$ is weak $*$ continuous on the dual space , and it follows that the range of this map, which the algebra numerical range, is convex.

4.2 Equality of sets

Numerical range is a set and set equality is a bi-conditional statement whose proof is done by showing set inclusion in both directions.

4.3 Algebraic Construction

The spectral properties of elementary operators have comprehensively been studied by many researchers.From the already established relationships between the spectrum of an elementary operator and the joint spectrum in the sense of Taylor of two commuting n-tuples A and B, it was possible to establish the obtained relations in terms of the numerical ranges. This approach has been adopted by researchers such as Seddik (2001, 2002, 2004); Kyle (1978); Shaw (1984) just to mention but a few. This is made possible by the fact that the spectrum of an operator is included in the closure of the numerical range and it would be natural to investigate if known facts pertaining the spectrum translate correspondingly to numerical ranges.

Also, from the theory of Banach spaces , the famous Hahn Banach theorems that allow us to algebraically construct functionals in a subspace and we are guaranteed of an extension in the whole space under consideration. Mathematical tools such as the finite rank operators have also been used in the construction.

Chapter 5

RESULTS AND DISCUSSION

5.1 Properties of the Numerical Range

In this chapter we give results on some properties of the numerical range. In particular, we establish the relationship between the numerical range of the generalized derivation restricted to a norm ideal J and that of its implementing elements. We extend these results to the concept of algebraic maximal numerical range as follows.

Given a Banach algebra $\mathscr{A}$, $\mathscr{A}^*$ the dual of $\mathscr{A}$, $S(\mathscr{A}) = \{x \in \mathscr{A} : \|x\| = 1\}$, the unit sphere, and $x \in S(\mathscr{A})$, let $D(x, \mathscr{A}) = \{f \in \mathscr{A}^* : f(x) = 1 = \|f\|\}$.

The Hahn-Banach theorem guarantees that $D(x, \mathscr{A})$ is non empty for each $x \in S(\mathscr{A})$. The elements of $D(I, \mathscr{A}), I$, the identity in $\mathscr{A}$, are called normalized states or simply states. For $a \in \mathscr{A}$, and $x \in S(\mathscr{A})$, we define $V(x, a, \mathscr{A}) = \{f(ax) : f \in D(x, \mathscr{A})\}$.

The numerical range of a is the set $V(a, \mathscr{A}) = \bigcup\{V(x, a, \mathscr{A}) : x \in S(\mathscr{A})\}$.

Given a Banach space $\mathscr{H}$, we may consider the Banach algebra $\mathscr{A} = L(\mathscr{H})$ and define the algebraic spatial numerical range of A by:

$W(A; L(\mathscr{H})) = \{f(Ax) : f \in \mathscr{H}^*, x \in \mathscr{H}, and\ \|f\| = \|x\| = 1 = f(x)\}$.

We first give some basic properties of the numerical range .

Bonsall (1969), has shown that $V(a, \mathscr{A}) = V(I, a, \mathscr{A})$, and for each $a \in \mathscr{A}, V(a, \mathscr{A})$

is a compact convex subset of $\mathbb{C}$. We give a simple proof of the following proposition.

Proposition 5.1. *$V(x,a,\mathscr{A}) = \{f(ax) : f \in D(x,\mathscr{A})\}$ is convex.*

Proof. Let $\lambda_1, \lambda_2 \in V(x,a,\mathscr{A})$. Then there exist support functionals $f_1, f_2 \in D(x,\mathscr{A})$ such that $\lambda_1 = f_1(ax)$,$\lambda_2 = f_2(ax)$.

Define f on $D(x,\mathscr{A})$ by $f(ax) = tf_1(ax) + (1-t)f_2(ax), t \in (0,1)$. We need to show that $f \in D(I,\mathscr{A})$. Clearly f is linear and

$$\begin{aligned} |f(ax)| &= |tf_1(ax) + (1-t)f_2(ax)| \\ &\leq t\,|f_1(ax)| + (1-t)\,|f_2(ax)| \\ &\leq t\,\|f_1\|\,\|ax\| + (1-t)\,\|f_2\|\,\|ax\| \\ &= \|ax\| \end{aligned}$$

$\Rightarrow \|f\| \leq 1.$

Also, $f(x) = tf_1(x) + (1-t)f_2(x) = 1$

$\Rightarrow \|f\| \geq 1.$

Thus $f \in D(I,\mathscr{A})$ and hence $V(x,a,\mathscr{A})$ is convex. □

For $a \in \mathscr{A}$, we define the left multiplication operator $L_a : \mathscr{A} \to \mathscr{A}$ by $L_a(x) = ax, \forall x \in \mathscr{A}$. $\|L_a\| = sup\{\|ax\| : x \in \mathscr{A}, \|x\| \leq 1\}$.

L_a is a linear operator in $\mathscr{A}$ and also a bounded operator since

$\|L_a\| = sup\{\|ax\| : x \in \mathscr{A}, \|x\| \leq 1\} \leq \|a\|$.

Similarly, the right multiplication operator for $b \in \mathscr{A}$ is defined by ;

$R_b : \mathscr{A} \to \mathscr{A}, x \to xb$.

We note that for all $x \in \mathscr{A}$ and fixed $a, b \in \mathscr{A}$, $\Delta_{a,b}(x) = L_a(x) - R_b(x) = ax - xb$, is the generalized derivation induced by $a, b \in \mathscr{A}$.

$L_a(\mathscr{A})$ will denote the set of all left multiplication operators on the algebra $\mathscr{A}$ as a ranges on $\mathscr{A}$. This set, endowed with the sup norm, is a normed algebra .

Proposition 5.2. *$L_a(\mathscr{A})$ is a unital normed algebra*

Proof. $L_a(\mathscr{A})$ is clearly a subspace of $L(\mathscr{A})$ (the set of bounded linear operators on $\mathscr{A}$) when we define addition and scalar multiplication point-wise by:

$(L_a + L_b)(x) = L_a(x) + L_b(x)$,

$(\lambda L_a)(x) = \lambda L_a(x), \forall\ x \in \mathscr{A}, a \in \mathscr{A}\ fixed,\ \lambda \in \mathbb{C}$.

Moreover, multiplication can also be defined point-wise by:

$(L_a L_b)(x) = L_a(x)L_b(x) = (ax)(xb), \forall\ x \in \mathscr{A}, a, b\ fixed$. Also the operator norm is sub-multiplicative in that:

$$\begin{aligned}
\|L_a L_b\| &= sup\,\{\|(L_a L_b)(x)\| : x \in \mathscr{A}, \|x\| = 1\} \\
&= sup\,\{\|L_a(x)L_b(x)\| : x \in \mathscr{A}, \|x\| = 1\} \\
&\leq sup\,\{\|L_a(x)\|\}\, sup\,\{\|L_b(x)\|\} \\
&= \|L_a\|\,\|L_b\|\,.
\end{aligned}$$

$\mathscr{A}$ is unital and so there exists a unit e such that $ex = xe = x, \forall x \in \mathscr{A}$. The operator $L_e \in L_a(\mathscr{A})$ such that$L_e(x) = ex = x$ is an identity left multiplication operator on $L_a(\mathscr{A})$ and is of norm one. Thus, equipped with the operator norm and the above defined operations, $L_a(\mathscr{A})$ is a unital normed algebra. □

Remark. $\mathscr{A}$ is a unital normed algebra and if it is not, then it can be unitized by adjoining a unit to it as outlined in chapter 1.

A similar argument shows that $R_a(\mathscr{A})$, set of all right multiplication operators on the algebra $\mathscr{A}$ as a ranges on $\mathscr{A}$ is also a normed algebra.

The algebraic numerical range of $L_a \in L_a(\mathscr{A})$ is the non-empty set:

$V(L_a; L_a(\mathscr{A})) = \{f(L_a); f \in L_a(\mathscr{A})^*, f(L_e) = 1 = \|f\|\}$. In Bonsall and Duncan (1971), it is shown that for any Banach algebra $\mathscr{A}$, $\|L_a\| = \|a\| = \|R_a\|$ and that $V\,(a; \mathscr{A}) = V(L_a; L(\mathscr{A})) = V(R_a; L(\mathscr{A}))$, $L(\mathscr{A})$ the algebra of all bounded linear operators on $\mathscr{A}$.

Lemma 5.1. *For* $a \in \mathscr{A}, L_a \in L_a(\mathscr{A}), \|L_a\| = \|a\| = \|R_a\|$

Proof.

$$\begin{aligned}\|L_a\| &= sup\,\{\|L_a(x)\| : \|x\| = 1\}\\ &= sup\,\{\|ax\| : \|x\| = 1\}\\ &\leq \|a\|\,\|x\|\\ &\Rightarrow \|L_a\| \leq \|a\|\,. \end{aligned} \tag{5.1}$$

If $\mathscr{A}$ has unit e, we have $L_a(e) = ae = a$ which implies

$$\|a\| = \|L_a(e)\| \leq \|L_a\|\,\|e\| = \|L_a\| \Rightarrow \|L_a\| \geq \|a\|\,. \tag{5.2}$$

From (5.1) and (5.2) equality follows.

Similarly we obtain $\|R_a\| = \|a\|$. □

Lemma 5.2. *For $a \in \mathscr{A}, V\,(a;\mathscr{A}) = V(L_a; L(\mathscr{A})) = V(R_a; L(\mathscr{A}))$*

Proof. Let $\lambda \in V(a : \mathscr{A})$, Then there exists $f \in S(\mathscr{A})$ such that $f(a) = \lambda$.

Now define F on $L(\mathscr{A})$ by

$F(L_a) = f(ax)$, for all $L_a \in L(\mathscr{A})$.

Clearly F is linear since

$$\begin{aligned}F\,(\alpha L_a + \beta L_b) =& f\,(\alpha ax + \beta bx)\\ =& f(\alpha ax) + f(\beta bx)\\ =& \alpha f(ax) + \beta f(bx)\\ =& \alpha F(L_a) + \beta F(L_b), a, b \in \mathscr{A}, \alpha, \beta \in \mathbb{C}.\end{aligned}$$

F is also bounded since

$\|F(L_a)\| = sup\,\{\|f(ax)\|\} \leq \|f\|\,\|ax\| = c\,\|L_a\|$.

Also $F(L_e) = f(ex) = f(x) = 1$ and $\|F\| = 1$.

So F as defined is a positive linear functional on $\mathscr{A}$.

Remark. A linear functional F is positive if and only if F is bounded and $\|F\| =$

$F(e)$. Refer to Sakai (2012) and Murphy (2014).

Take a finite rank operator $b \in L(\mathscr{A})$ defined by

$bx = g(x)a$, for all $x \in \mathscr{A}, g \in S(\mathscr{A})$. Clearly $\|b\| = 1$ and $F(b) = f(bx) = f(g(x)a) = g(x)f(a) = \lambda$. Hence $V(a; \mathscr{A}) \subseteq V(L_a; L(\mathscr{A})$.

Conversely we show that $V(L_a; L(\mathscr{A})) \subseteq V(a; \mathscr{A})$.

Let $\lambda \in V(L_a; L(\mathscr{A}))$. Then there exists a state $f \in L(\mathscr{A})^*$ such that $f(L_a) = \lambda$.

Define a functional $h \in \mathscr{A}^*$ by $h(a) = f(L_a)$.

Then

$$\begin{aligned} h(\alpha a + \beta b) =& f(\alpha L_a + \beta L_b) \\ =& f(\alpha L_a) + f(\beta L_b) \\ =& \alpha f(L_a) + \beta f(L_b) \\ =& \alpha h(a) + \beta h(b). \end{aligned}$$

$\Rightarrow h$ is linear and bounded. $h \in \mathscr{A}^*$ is also positive since $h(a^*a) = f(L_a^* L_a) \geq 0$.

Furthermore h is of norm 1 since $h(e) = f(L_e) = 1$ and

$1 = |h(e)| \leq \|h\| \, \|e\| \Rightarrow \|h\| \geq 1$. We also have

$$\begin{aligned} \|h\| &= sup\{|h(a)| : \|a\| = 1\} \\ &= sup\{|f(L_a)| : \|L_a| = 1\} \\ &\leq \|f\| \\ &= 1. \end{aligned}$$

Thus h is a state on $\mathscr{A}^*$ and so $V(L_a; L(\mathscr{A})) \subseteq V(a; \mathscr{A})$. $\square$

5.2 Norms of R_A and R_B in Ideals

Let X and Y be Banach algebras. $L(X)$ and $L(Y)$, the algebra of all bounded linear operators on X and Y respectively. Let $(J, \|.\|_J)$ be a norm ideal on $L(Y, X)$, the algebra of all bounded linear operators from Y to X such that:

i) $(J, \|.\|_J)$ is a Banach space

ii) If $A \in L(X), T \in J, B \in L(Y)$ then $ATB \in J$,and $\|ATB\|_J \leq \|A\| \, \|T\|_J \, \|B\|$

iii) $\|T\| \leq \|T\|_J$, $T \in J$ and

iv) $\|T\|_J = \|T\|$, for T a rank- one operator.

If $A \in L(X), B \in L(Y)$ and $T \in J$, then the operators L_A, R_B and $L_A - R_B$ are all bounded and linear on $L(J)$, the space of all bounded linear operators from J to J, where:
$L_A T = AT$, the left multiplication operator,
$R_B T = TB$, the right multiplication operator and
$(L_A - R_B) T = AT - TB$, the generalized derivation.The following result holds.

Theorem 5.1. *$V(A : L(X)) = V(L_A : L(J))$.*

Proof. Let $\lambda \in V(A : L(X))$. Then there exist $f \in L(X)^*$ such that
$\lambda = f(A), and, f(I_{L(X)}) = 1 = \|f\|$.
Let $\mathscr{A}_0 = \{L_A : A \in L(X), L_A(T) = AT, T \in J\} \subseteq L(J)$.
$\mathscr{A}_0$ is a linear subspace of $L\left(L(X)\right)$.
On $\mathscr{A}_0^*$, define a linear functional g such that $g(L_A) = f(A)$. Clearly g as defined is a state and the Hahn-Banach theorem guarantees the existence of its extension on $L(J)$. Hence, $V(A : L(X)) \subseteq V(L_A : L(J))$.
Conversely, suppose $\lambda \in V(L_A : L(J)$. Then there exists $f \in L(J)^*$ such that $f(L_A) = \lambda$ and $f(I_{L(J)}) = 1 = \|f\|$.
Define a linear operator h on $L(X)^*$ by $h(A) = f(L_A)$. Then $h(I) = f(I_{L(J)}) = 1$.
h is thus a state on $L(X)^*$ and $V(L_A : L(J)) \subseteq V(A : L(X))$. □

Theorem 5.2. *$\|L_A\|_J = \|A\|$.*

Proof. From the definition of a norm ideal, we infer that L_A and R_B are bounded linear operators on $(J. \|.\|_J)$ and

$$\begin{aligned}\|L_A\|_J &= Sup\{\|AX\| : \|X\|_J = 1, X \in J\}\\ &\leq \|A\| \, \|X\|_J\\ &= \|A\| \, .\end{aligned}$$

Also, from the definition of a norm ideal, we know that $\|L_A\|_J \geq \|A\|$.
It therefore follows that $\|L_A\|_J = \|A\|$.
Similarly $\|R_B\|_J = \|B\|$. □

5.3 Numerical Range of the Generalized Derivation in a Norm Ideal

In the past, generalized derivations, their properties and restrictions to norm ideals have been investigated by many authors. For example, their spectra have been characterized by Fialkow (1979, 1980). The famous results on the norms of inner derivation and the generalized derivation as obtained by Stampfli (1970) using maximal numerical range have since provided a crucial lead in defining norms of elementary operators. We recall the works of Kyle (1978) who examines the relationship between the numerical range of an inner derivation, and that of its implementing element. Using the already established results involving the spectrum, that is, for any Banach algebra $\sigma(\Delta_A) = \sigma A - \sigma A$, Kyle obtained the corresponding result for numerical ranges.
Magajna (1987) gave the essential numerical range of the generalized derivation defined on the Hilbert-Schmidt class in terms of the numerical and the essential numerical ranges of the implementing operators. Shaw (1984), in particular, established that the algebra numerical range of a generalized derivation restricted to a norm ideal J is equal to the difference of the algebra numerical ranges of the implementing operators provided that J contains all finite rank operators and

is suitably normed. With slight modification we obtain an alternative proof to Shaw's result.

Theorem 5.3. *Let J be a normed ideal on the algebra $L(Y,X)$. Then for $A \in L(X), B \in L(Y), V(\Delta_{A,B} : L(J)) = V(A : L(X)) - V(B : L(Y))$.*

Proof. Let $\lambda \in V(\Delta_{A,B} : L(J))$. This implies there exists $f \in L(J)^*$ such that $f(\Delta_{A,B}) = \lambda$ and $f\left(I_{L(J)}\right) = 1 = \|f\|$. Let $\mathscr{A}_0 = \{L_A : A \in L(X), L_A(T) = AT, T \in J\} \subseteq L(J)$ and

$\mathscr{A}_1 = \{R_B : B \in L(Y), R_B(T) = TB, T \in J\} \subseteq L(J)$, that is, the set of the left and right multiplication operators respectively in $L(J)$. These are linear subspaces of $L(X)$ and $L(Y)$ respectively. Let also $S(L(J)) = \left\{f \in L(J)^* : f\left(I_{L(J)}\right) = 1 = \|f\|\right\}$, then

$$\begin{aligned}
\lambda = f(\Delta_{A,B} : L(J)) &= \{f(L_A - R_B : f \in S(L(J)))\} \\
&= \left\{f(L_A) : f \in L(X)^*, f(I_{L(X)}) = 1 = \|f\|\right\} \\
&- \left\{f(R_B) : f \in L(Y)^*, f(I_{L(Y)}) = 1 = \|f\|\right\} \\
&= V(L_A : L_A \in L(J)) - V(R_B : R_B \in L(J)).
\end{aligned}$$

This implies that $\lambda \in V(A : L(X)) - V(B : L(Y))$.

To prove the reverse inclusion, we make use of the spatial numerical range. Choose λ in $W(A : L(X))$ and μ in $W(B : L(Y))$. Then we can find functionals f and g in $L(X)^*, L(Y)^*$ such that

$\|f\| = \|x\| = f(x) = 1$, with $f(Ax) = \lambda$ and

$\|g\| = \|y\| = g(y) = 1$, with $g(By) = \mu$.

Let X be a rank one operator in J such that $Xz = g(z)x$, for all $z \in Y$.

Also define F in $L(J)^*$ by $F(T) = f(Ty)$, for all $T \in L(J)$.

Then $F(X) = f(Xy) = fg(y)x = g(y)f(x) = 1$,

$F(I) = f(Iy) = fg(y)x = g(y)f(x) = 1$ and

$|F(T)| \leq \|f\| \, \|T\|_J \, \|y\| = \|T\|_J$.

Clearly $\|F\|_J = \|X\|_J = 1$ and $\left(I_{L(J)}, F\right) \in L(J) \times L(J)^*$.

Thus,

$$\begin{aligned}
F(\Delta_{A,B}(X)) &= F\left(AX - XB\right) \\
&= f\left(AX - XB\right)y \\
&= f(AXy) - f(XBy) \\
&= f(g(y)Ax) - f(g(By)x) \\
&= f(Ax)g(y) - f(x)g(By) \\
&= \lambda - \mu \\
&\in \left\{W\left(A : L(X)\right) - W\left(B : L(Y)\right)\right\}.
\end{aligned}$$

Now

$$\begin{aligned}
V\left(\Delta_{A,B}; L(J)\right) &= \overline{co}W\left(\Delta_{A,B}; L(J)\right) \\
&\supseteq \overline{co}\left\{W\left(A; L(X)\right) - W\left(B; L(Y)\right)\right\} \\
&= \overline{co}\left\{W\left(A; L(X)\right)\right\} - \overline{co}\left\{W\left(B; L(Y)\right)\right\} \\
&= V\left(A; L(X)\right) - V\left(B; L(Y)\right).
\end{aligned}$$

Thus $\{V\left(A; L(X)\right) - V(B; L(Y))\} \subseteq V(\Delta_{A,B}; L(J))$. □

The next result gives the upper bound of the norm of $L_A - R_B$ in $(J, \|.\|_J)$.

Theorem 5.4. $\|L_A - R_B\|_J \leq \|A - \lambda\| + \|B - \lambda\|$.

Proof.

$$\begin{aligned}
\|L_A - R_B\|_J = \|L_{A-\lambda} - R_{B-\lambda}\|_J &= \|AX - \lambda X - XB + X\lambda\|_J \\
&= \|\left(A - \lambda\right)X - X\left(B - \lambda\right)\|_J \\
&\leq \left(\|A - \lambda\| + \|B - \lambda\|\right)\|X\|_J \\
&= \|A - \lambda\| + \|B - \lambda\|.
\end{aligned}$$

□

5.4 Maximal Numerical Range of Elementary Operators

If A and B are n-tuples of commuting operators on $H, W(A), W(B)$ the usual numerical ranges of A and B, $V(R_{A,B})$ the algebraic numerical range of $R_{A,B}$, then:

(i) Seddik (2001) establishes that the numerical range of an elementary operator acting on the Banach space of the p-Schatten class operators on H,$(\mathscr{C}_p(H), \|.\|_p)$,for $p \geq 1$, satisfies the relation $co(W(A)\circ W(B) \subset V(R_p(A, B))$.

(ii) In Seddik (2002) it is proved that $co(W(A) \circ W(B) \subset V(R_{A,B})$ without the assumption of the commutativity.

Here we establish results using the maximal numerical range. For vectors x and y in a Hilbert space H, define a finite rank operator by $(x \otimes y)z = \langle z, y\rangle\, x, \quad \forall z \in H$. We take $S \subseteq L(H)$ to be an operator algebra containing finite rank operators.
For a set M, we denote by $\overline{M}$ and coM the closure and the convex hull of M respectively.

Theorem 5.5. *Let A and B be two n-tuples of commuting operators on H, $W_\circ(A)$ and $W_\circ(B)$, the maximal numerical ranges of A and B, and $V_\circ(R_{A,B} \mid S)$, the algebraic maximal numerical range of the elementary operator restricted on S. Then*
$co\overline{(W_\circ(A) \circ W_\circ(B))} \subset V_\circ(R_{A,B} \mid S)$.

Proof. Let $\lambda \in W_\circ(A), \beta \in W_\circ(B)$.
This implies there exists sequences $\{x_n\}, \{y_n\} \in H$ such that $\|x_n\| = \|y_n\| = 1$,

$$\lim_{n\to\infty} \{\langle A_1 x_n, x_n\rangle, ..., \langle A_n x_n, x_n\rangle\} = \{\lambda_1, ..., \lambda_n\} = \lambda,\ \|A_i x_n\| \to \|A_i\|$$

and

$$\lim_{n\to\infty} \{\langle B_1 x_n, x_n\rangle, ..., \langle B_n x_n, x_n\rangle\} = \{\beta_1, ..., \beta_n\} = \beta, \ \|B_i x_n\| \to \|B_i\|, 1 \leq i \leq n.$$

Define f on $L(S)$ by

$$f(\Omega) = \lim_{n\to\infty} \langle \Omega(x_n \otimes y)z_n, \ (x_n \otimes y_n)z_n\rangle, \forall \ \Omega \in L(S).$$

Clearly ,

$$\begin{aligned}\|x_n \otimes y_n\| &= \sup\{\|(x_n \otimes y_n)z_n\| : z_n \in H, \|z_n\| = 1\}\\ &= \|x_n\| \|y_n\|\\ &= 1.\end{aligned}$$

Also ,

$$\begin{aligned}|f(\Omega)| &= |\langle \Omega(x_n \otimes y_n)z_n, (x_n \otimes y_n)z_n\rangle|\\ &= |\langle z_n, y_n\rangle|^2 |\langle \Omega x_n, x_n\rangle|\\ &\leq \|z_n\|^2 \|y_n\|^2 \|\Omega x_n\| \|x_n\|\\ &\leq \|z_n\|^2 \|y_n\|^2 \|\Omega\| \|x_n\|^2\\ &= \|\Omega\|\\ &\Longrightarrow |f(\Omega)| \leq \|\Omega\|\end{aligned}$$

and so f is bounded and $\|f\| \leq 1$. Assume $I \in S$ (S can be unitized in case its non-unital) then,

$$\begin{aligned}
f(I) &= lim \left\langle (x_n \otimes y_n)z_n, (x_n \otimes y_n)z_n \right\rangle \\
&= lim \left\langle \langle z_n, y_n \rangle x_n, \langle z_n, y_n \rangle x_n \right\rangle \\
&= lim \left| \langle z_n, y_n \rangle \right|^2 \|x_n\|^2 \\
&= \|z_n\|^2 \|y_n\|^2 \|x_n\|^2 \\
&= 1.
\end{aligned}$$

$f(I) = 1$ so that $\|f(I)\| = 1 \leq \|f\| \Longrightarrow \|f\| \geq 1$ Thus $\|f\| = 1$.

A linear functional f is said to be positive if $f(\omega\omega^*) \geq 0$ for all $\omega \in S$. Taking a sequence of unit vectors z in H we see that f is a positive linear functional since

$$\begin{aligned}
f(\Omega\Omega^*) &= \langle \Omega\Omega^*(x_n \otimes y_n)z_n, (x_n \otimes y_n)z_n \rangle \\
&= \langle \Omega^*(x_n \otimes y_n)z_n, \Omega^*(x_n \otimes y_n)z_n \rangle \\
&= \|\Omega^*(x_n \otimes y_n)z_n\|^2 \geq 0.
\end{aligned}$$

Since f is a positive linear functional of unit norm, it follows that f is a state on $L(S)$. Moreover, f is clearly a maximal state. Recall now that

$$R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i = A_1 X B_1 + A_2 X B_2 + \cdots + A_n X B_n, \forall\ X \in L(H).$$

Thus for $X \in S$ we have,

$$\begin{aligned} f\left(\sum_{i=1}^{n} A_i X B_i\right) &= f\left(\sum_{i=1}^{n} A_i (x_n \otimes y_n) B_i z_n\right) \\ &= f\left(\sum_{i=1}^{n} A_i x_n \langle B_i z_n, y_n\rangle\right) \\ &= \sum_{i=1}^{n} \left(f(A_i x_n) \langle B_i z_n, y_n\rangle\right) \\ &= \sum_{i=1}^{n} f(A_i x_n) g(B_i y_n), \quad \forall z_n = y_n \\ &= \sum_{i=1}^{n} \lambda_i \beta_i \in V_\circ(R_{A,B} \mid_S). \end{aligned}$$

This shows that $W_\circ(A) \circ W_\circ(B) \subset V_\circ(R_{A,B} \mid S)$ and since the algebraic maximal numerical range is compact and convex, we deduce:
$co\overline{(W_\circ(A) \circ W_\circ(B))} \subset V_\circ(R_{A,B} \mid S)$. □

Corollary 5.1. *Let $A \in L(H)$. Then $V_\circ(L_A) = V_\circ(R_A) = V_\circ(A)$.*

Proof. If A is an operator on a Hilbert space H, Fong (1979) has showed that $V_\circ(A) = W_\circ(A)$. The inclusion $V_\circ(A) \subseteq V_\circ(L_A)$ follows from this and the above theorem 5.5.
Now, let $\lambda \in V_\circ(L_A)$. Then there exists f in $(L(L_A))^*$, the algebra of all bounded operators on the set of left multiplication operators, such that
$f(L_A) = \lambda, f(I) = 1 = \|f\|$ and $f(L_A^* L_A) = \|L_A\|^2$.
Define a functional g on $L(H)$ by $g(A) = f(L_A)$. By simple computation, we see that g is a maximal state on $L(H)$ so that $g(A) = f(L_A) \in V_\circ(A)$. Therefore $V_\circ(L_A) \subseteq V_\circ(A)$. By the same argument, we find also that $V_\circ(R_A) = V_\circ(A)$. □

Corollary 5.2. *For $A, B \in L(H), V_\circ(A) - V_\circ(B) = V_\circ(\Delta_{A,B})$.*

Proof. By theorem 5.5, we have $W_\circ(A) - W_\circ(B) \subseteq V_\circ(\Delta_{A,B})$ and since $V_\circ(\Delta_{A,B})$ is closed, then we have
$\overline{(W_\circ(A) - W_\circ(B))} = V_\circ(A) - V_\circ(B) \subseteq V_\circ(\Delta_{A,B})$.
For the reverse inclusion, we apply the properties of the numerical range as stated

in chapter 3.

$$
\begin{aligned}
V_\circ\left(\Delta_{A,B}, \mathscr{A}\right) &= \left\{f\left(\Delta_{A,B}\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\right\} \\
&= \left\{f\left(L_A - R_B\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\right\} \\
&\subseteq \left\{f\left(L_A\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\right\} - \left\{f\left(R_B\right) : f \in S_\circ\left(L\left(\mathscr{A}\right)\right)\right\} \\
&= V_\circ\left(L_A\right) - V_\circ\left(R_B\right) \\
&= V_\circ(A) - V_\circ(B),
\end{aligned}
$$

so that $V_\circ\left(\Delta_{A,B}, \mathscr{A}\right) \subseteq V_\circ(A) - V_\circ(B)$. $\square$

5.5 Derivation in the quotient Algebra

Let H be a separable infinite dimensional Hilbert space. $L(H)$, the algebra of bounded linear maps on H. $L(H)$ is a C^*- algebra. The algebra $K(H)$ of all compact operators acting on the Hilbert space H is a norm closed sub-algebra of $L(H)$. $K(H)$ is a two sided closed ideal of $L(H)$ and closed under involutions; hence it is also a C^*- algebra. Since $K(H)$ is a closed ideal in $L(H)$ we may form the quotient Banach algebra $\dfrac{L(H)}{K(H)}$ called the Calkin algebra. Actually it is also a C^*- algebra and in particular a simple one. For $[t] = T + K(H) \in \dfrac{L(H)}{K(H)}$, we define the essential norm by

$$\|T\|_e = \|T + K\ (H)\| = inf\left\{\|T + K\| : K \in K(H)\right\}.$$

The essential numerical range of an operator $T \in L(H)$, denoted $W_e(T)$ was defined by Stampfli and Williams (1968) as the numerical range of the coset $[t] = T + K(H)$ in the Calkin algebra.They proved the following equality and it has been adopted as the definition of the essential numerical range:
$W_e(T) = \cap \left\{\overline{W(T + K)}, K \in K(H)\right\}$.
Some of the basic properties of the essential numerical range include:

i) $W_e(T)$ is a non-void, compact and convex set.

ii) $W_e(T) \subseteq \overline{W(T)}$

iii) $\sigma_e(T) \subset W_e(T)$, $\sigma_e(T)$ the essential spectrum of T.

iv) $W_e(T) = \{0\}$ iff $T \in K(H)$

The algebra essential numerical range is given by

$$V_e(T) = \left\{f(T + K) : f \in \left(\frac{L(H)}{K(H)}\right)^*, f\,(I + K) = 1 = \|f\|\right\}$$
$$= \left\{f(T) : f \in L(H)^*, f(I) = 1 = \|f\|\,, f\,(K(H)) = 0\right\}.$$

Now, for all $[t] \in L(H)/K(H)$, we define the following operators on $\dfrac{L(H)}{K(H)}$:

a) The left and right multiplication operators,

$L_{[a]}\,[t] = [a]\,[t]$ and $R_{[a]}\,[t] = [t]\,[a]$, $[a] \in \dfrac{L(H)}{K(H)}$ fixed, respectively.

b) The inner derivation, $\Delta_{[a]}\,[t] = [a]\,[t] - [t]\,[a]\,,\ [a] \in \dfrac{L(H)}{K(H)}$ fixed.

c) The outer derivation, $\Delta_{[a],[b]}\,[t] = [a]\,[t] - [t]\,[b]\,,\ [b] \in \dfrac{L(H)}{K(H)}$ fixed.

Following the approach of the work done by Bonsall (1969), let $S\left(\dfrac{L(H)}{K(H)}\right), \left(\dfrac{L(H)}{K(H)}\right)^*$ and $L\left(\dfrac{L(H)}{K(H)}\right)$ denote respectively the unit sphere $\{[x] : \|x\|_e = 1\}$ of $\dfrac{L(H)}{K(H)}$, the dual space of $\dfrac{L(H)}{K(H)}$ and the set of all linear mappings of $\dfrac{L(H)}{K(H)}$ into $\dfrac{L(H)}{K(H)}$. For each $[x] \in S\left(\dfrac{L(H)}{K(H)}\right)$ and $\Gamma \in L\left(\dfrac{L(H)}{K(H)}\right)$, let
$D([x]\,, \dfrac{L(H)}{K(H)}) = \left\{ f \in \left(\dfrac{L(H)}{K(H)}\right)^* : \|f\| = f\,[x] = 1 \right\}$ and
$V\left(\Gamma : [x]\right) = \left\{ f\left(\Gamma\,[x]\right) : f \in D([x]\,, \dfrac{L(H)}{K(H)}) \right\}$.
The numerical range $V(\Gamma)$ is defined by $V(\Gamma) = \cup \left\{ V\left(\Gamma : [x]\right) : [x] \in S\left(\dfrac{L(H)}{K(H)}\right) \right\}$.
Using the above definition, the numerical range of the left multiplication operator will be given by

$$\begin{aligned} V\left(L_{[a]}\right) &= \cup \left\{ V\left(L_{[a]} : [x]\right) : [x] \in S\left(L(H)/K(H)\right) \right\} \\ &= \cup \left\{ f\left(L_{[a]}\,[x]\right) : f \in D([x]\,, L(H)/K(H)) \right\} \\ &= \cup \left\{ f\left([a]\,[x]\right) : f \in D([x]\,, L(H)/K(H)) \right\} \\ &= \cup \left\{ f\left([ax]\right) : f \in D([x]\,, L(H)/K(H)) \right\}. \end{aligned}$$

Similarly, the numerical range of the right multiplication operator can be obtained. Recall, the algebraic maximal numerical range of an element a in a C*-algebra $\mathscr{A}$, denoted by $V_o(a, \mathscr{A})$ is defined to be the set:

$$\{f(a) : f \in S_0(a, \mathscr{A})\},$$

where $S_0(a, \mathscr{A})$ denotes the set of all maximal states of a.

Consider the general operator $\Gamma_{[a],[b]}([x]) = [a]\,[x] + [x]\,[b] = L_{[a]} + R_{[b]}, \forall\,[x] \in \mathscr{A} = L(H)/K(H)$.

Theorem 5.6. *For* $[a]\,,[b]$ *in the C*-algebra* $\mathscr{A} = L(H)/K(H)$*,we have* $V_o(\Gamma_{[a],[b]}, \mathscr{A}) \subseteq V_o(\Gamma_{[a]}, \mathscr{A}) + V_o(\Gamma_{[b]}, \mathscr{A})$.

Proof.

$$\begin{aligned}
V_o(\Gamma_{[a],[b]}, \mathscr{A}) &= \left\{f(\Gamma_{[a],[b]}) : f \in S_0(\Gamma_{[a],[b]}, \mathscr{A})\right\} \\
&= \left\{f(L_{[a]} + L_{[b]}) : f \in S_0(\Gamma_{[a],[b]}, \mathscr{A})\right\} \\
&= \left\{f(L_{[a]}) + f(L_{[b]}) : f \in S_0(\Gamma_{[a],[b]}, \mathscr{A})\right\} \\
&\subseteq \left\{f\left(L_{[a]}\right) : f \in S_0(L_{[a]}, \mathscr{A})\right\} + \left\{f\left(R_{[b]}\right) : f \in S_0(R_{[b]}, \mathscr{A})\right\} \\
&= V_o(\Gamma_{[a]}, \mathscr{A}) + V_o(\Gamma_{[b]}, \mathscr{A}).
\end{aligned}$$

$\square$

Theorem 5.7. *For the inner derivation* $\Delta_{[t]}\,[a] = [t]\,[a] - [a]\,[t]\,, \forall\,[a] \in \mathscr{A}$*,* $V_o([t]\,[a] - [a]\,[t]\,, \mathscr{A}) \subseteq V_o([t]\,[a]\,, \mathscr{A}) - V_o([a]\,[t]\,, \mathscr{A})$.

Proof.

$$\begin{aligned}
V_o([t]\,[a] - [a]\,[t]\,, \mathscr{A}) &= \{f([t]\,[a] - [a]\,[t]) : f \in S_0(\mathscr{L}(\mathscr{A}))\} \\
&\subseteq \{f([t]\,[a]) : f \in S_0(\mathscr{L}(\mathscr{A}))\} - \{f([a]\,[t]) : f \in S_0(\mathscr{L}(\mathscr{A}))\} \\
&= V_o([t]\,[a]\,, \mathscr{A}) - V_o([a]\,[t]\,, \mathscr{A}).
\end{aligned}$$

$\square$

We now derive the upper bound for the norm of an inner derivation in the quotient C* algebra.

Let $\pi : L(H) \to L(H)/K(H)$ be the canonical homomorphism from $L(H)$ such that $T \to T + K$. π is continuous. Let $\Delta_T : L(H) \to L(H)$ be the inner derivation map such that $X \to TX - XT, \forall X \in L(H)$. The map Δ_T, is also continuous .

The composite map $\pi \circ \Delta_T$ is continuous and induces the derivation map $\hat{\Delta}_{[t]} : L(H)/K(H) \to L(H)/K(H)$ such that $[x] \to [t]\,[x]-[x]\,[t]\,, \forall\,[x] \in L(H)/K(H)$.

Theorem 5.8. *Let* $\hat{\Delta}_{[t]} = [tx - xt]$*, for all* $[t]\,,[x] \in L(H)/K(H)$*. Then there exists* $\lambda \in \mathbb{C}$ *such that* $\left\|\hat{\Delta}_{[t]}\,[x]\right\| \leq 2inf\,\{\|[t-\lambda]\| : \lambda \in \mathbb{C}\}/$

Proof. Notice that $\hat{\Delta}_{[t]} = [tx - xt] = \Delta_T + K(H)$.

By definition,

$$\left\|\hat{\Delta}_{[t]}\right\| = sup\,\{\|[t]\,[x] - [x]\,[t]\| : [x] \in L(H)/K(H), \|[x]\| = 1\}\,.$$

However,

$$\begin{aligned}
\left\|\hat{\Delta}_{[t]}\,[x]\right\| &= \|[t]\,[x] - [x]\,[t]\| \\
&= \|[tx] - [xt]\| \\
&= \|[tx] - [\lambda x] + [\lambda x] - [xt]\| \\
&= \|[tx - \lambda x] - [xt - \lambda x]\| \\
&= \|[(t-\lambda)\,x] - [x\,(t-\lambda)]\| \\
&= \|[t-\lambda]\,[x] - [x]\,[t-\lambda]\| \\
&\leq 2\,\|[t-\lambda]\|\,\|[x]\|\,.
\end{aligned}$$

Taking the supremum over all $[x]$ of norm one yields,

$$\left\|\hat{\Delta}_{[t]}\right\| \leq 2inf\,\{\|[t-\lambda]\| : \lambda \quad complex\}\,.$$

$\square$

Chapter 6

CONCLUSION AND RECOMMENDATIONS

6.1 Conclusion

In the study, the convexity of the algebra numerical range has been proved. The algebra numerical range of a generalized derivation restricted to a norm ideal J has been shown to be equal to the difference of the algebra numerical ranges of the implementing operators provided that J contains all finite rank operators and is suitably normed . Furthermore, the relationship that exists between the algebraic maximal numerical range of an elementary operator and its implementing operators has been established .

6.2 Recommendations

The inclusion $co\overline{(W_\circ(A) \circ W_\circ(B))} \subset V_\circ(R_{A,B} \mid S)$ has been shown in one direction and it is natural to wonder if the reverse inequality or equality holds true.

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